Spherically Symmetric Non Linear Structures

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We present an analytical method to extract observational predictions about non linear evolution of perturbations in a Tolman Universe. We assume no a priori profile for them. We solve perturbatively a Hamilton - Jacobi equation for a timelike geodesic and obtain the null one as a limiting case in two situations: for an observer located in the center of symmetry and for a non-centered one. In the first case we find expressions to evaluate the density contrast and the number count and luminosity distance vs redshift relationships up to second order in the perturbations. In the second situation we calculate the CMBR anisotropies at large angular scales produced by the density contrast and by the asymmetry of the observer's location, up to first order in the perturbations. We develop our argument in such a way that the formulae are valid for any shape of the primordial spectrum.

I. INTRODUCTION

The study of large - scale structure formation represents one of the most exciting research fields in Cosmology. The currently accepted view is that the structures we observe today trace their origin to primeval density inhomogeneities, generated by zero point fluctuations in the scalar field responsible for Inflation. The evolution of these inhomogeneities, from certain initial values ($<< 1$) to their present distribution in stars, galaxies, clusters of galaxies and so on, involves complex non linear hydrodynamic and gravitational processes. Due to this complexity, it has not been possible to study this evolution from a completely analytical perspective [1], [2], [3].

Given the impossibility at present (and possibly in principle [4]) to find a general exact analytical solution to Einstein equations, our understanding of processes such as the clustering of galaxies is largely dependent on numerical techniques. Of the known exact solutions to those equations, most of them presuppose the presence of certain symmetries of space-time and/or energy density [5], [6]. One of them is the Tolman solution, which describes an inhomogeneous Universe filled with pressureless matter and spherically symmetric around a point. This solution is completely characterized by two time independent functions $f^2(r)$ and $F(r)$. This last one can be interpreted as the mass contained in a sphere and $f^2(r)$ as the mechanical energy of a shell, both of radius r.

In spite of its limitations, the Tolman solution has been fruitfully used to study a great variety of effects related to the formation of large scale structures: formation of large scale voids, anisotropies in the microwave background radiation, possible fractal distribution of galaxies, etc. [7], [8], [9], [10], [11], [12] [13]. But all (or almost all) of these studies are based on the numerical integration of the corresponding equations (mainly the equation for null geodesics), and often simplified expressions for $f^2(r)$ and $F(r)$ are employed, sometimes with no other reason than to facilitate the numerical calculations.

The goal of this paper is to present an analytical method to study the non linear evolution of density perturbations in a matter dominated Universe, viewed as an instance of the Tolman Universe, laying the emphasis in a physically motivated choice for the Tolman functions. Concretely, we shall assume the Tolman functions to be such that at early times, in the linear regime, the model be equivalent to a spatially flat Friedmann - Robertson - Walker (FRW) matter dominated Universe with growing perturbations. Otherwise,

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we shall keep the Tolman functions completely general. Observe that in this way the Tolman functions are determined by the spectrum of perturbations to the original FRW model.

We shall focus on simulating in our spherically symmetric model the results of concrete cosmological observations such as the redshift - luminosity distance and the number count - redshift relationships. Since these observations rely on information carried by light, the analysis centers on the study of the null geodesics in the model. We device a perturbative method based on the Hamilton - Jacobi equation for a time like geodesic and obtain the null geodesic as a limiting case.

We apply our scheme to study two situations. In the first one we consider an observer located in the center of the Universe and evaluate the number count and luminosity distance vs redshift relationships. For the first cosmological test we assume the simplifying hypothesis that the luminosity of galaxies does not evolve, which is justified in view of the fact that the redshifts we consider are small ($z < 0.08$). For larger redshifts however, the evolution of luminosity ought to be taken into account [14].

In the second situation we consider an observer located away from the center and study the anisotropies in the cosmic background radiation due to the presence of perturbations in the photon path, from the last scattering surface to the observer's position.

Throughout the paper a unique matter component of the Universe is assumed.

We begin our discussion by briefly reviewing the theory of linearized scalar adiabatic perturbations to a FRW background, the main features of the Tolman solution to the Einstein equations and the matching procedure. In section **III** we review the cosmological observations N vs z and d_ℓ vs z and write their form for a Tolman Universe. In section **IV** we begin by sketching our method, we perform the explicit calculations for a centered observer and find the general expression for the mentioned observational tests. In section **V** we do the same as in section **IV** but for a non centered observer and evaluate the anisotropies in the CMBR temperature due to the different paths of the photon from the last scattering surface to the observer's position. To obtain an idea of the kind of results to be expected from concrete perturbation profiles, in section **VI** we apply our formulae to a particular Tolman Universe, corresponding to a scale invariant spectrum of perturbations. Finally we discuss our results in section **VII**.

The overall conclusion is that the method we propose affords a simple way to model the growth of nonlinear structures such as large scale voids, and may be used profitably in testing competing theories of primordial fluctuation generation.

II. MATCHING A FRW UNIVERSE WITH PERTURBATIONS TO A TOLMAN SOLUTION

As stated in the Introduction, the accepted framework for structure formation consists in non linear growth of perturbations during the matter dominated era of the Universe. The origin of these perturbations are the zero point fluctuations of the scalar field responsible for driving Inflation. As the Universe inflates, the wavelengths of the modes that build fluctuation grow and become bigger than the Hubble length, leaving the region where microphysics can act. At this stage, the amplitude of each mode is frozen and so remains until the mode reenters the Hubble sphere. As long as a mode is outside the horizon its evolution can be studied within linear theory. We therefore can have an accurate prediction of the amplitudes of all those modes whose wavelengths are equal to or bigger than the Hubble length of the considered epoch. Substantial growth, moreover, occurs only during the matter dominated epoch of the Universe (modes that reenter the horizon during the radiation dominated epoch grow logarithmically at most [15]).

In this context, we will characterize the Tolman Universe by matching it to a flat Friedmann Robertson Walker Universe with perturbations, at the epoch of equilibrium between the matter and radiation energy densities.

To simplify the matching procedure we only consider modes bigger than (or at most equal to) the Hubble size at that epoch. This is certainly justified in hot dark matter models, where smaller fluctuations are washed out by Landau damping and free streaming; in more general models a consideration of the transfer function would be called for. We shall neglect decaying perturbations to a FRW model, but otherwise do not assume any particular spectrum for the perturbations.

We shall also assume that light traces matter, i.e. a Universe with only one matter component.

A. Scalar Perturbations in a Flat FRW Universe

The most general metric tensor that describes spherically symmetric scalar perturbations in a spatially flat FRW Universe is [16]

$$\delta g_{\mu\nu} = \begin{pmatrix} 2\phi & -aB' & 0 & 0 \\ -aB' & 2a^2\left[\psi - E_{|rr}\right] & 0 & 0 \\ 0 & 0 & 2a^2\left[\psi - E_{|\theta\theta}\right] & 0 \\ 0 & 0 & 0 & 2a^2\left[\psi - E_{|\varphi\varphi}\right]\sin^2\theta \end{pmatrix} \tag{1}$$

where ϕ, ψ, B, E are scalar functions of r and t, a prime means partial derivative with respect to r, a dot means partial derivative with respect to the cosmic time t and a bar means spatial covariant derivative. $a = (t/t_0)^{2/3}$ is the scale factor for an unperturbed FRW Universe, where the density is critical, $\Omega_0 = 1$.

We choose t_0 as the age of the Universe at the time of equilibrium between the matter and radiation energy contents of the Universe ($(1+z)_{eq} = 2.32 \times 10^4$), $t_0 \simeq 5.65 \times 10^{-4} h^{-1} Mpc$ [17] (the speed of light, $c = 1$, and h is the fudge factor for the Hubble constant, $H = 100h\ kms^{-1}Mpc^{-1} \rightarrow 3 \times 10^{-4} h\ Mpc^{-1}$ in natural units).

The theory of linearized scalar perturbations is treated extensively in Ref. [16]. In the longitudinal gauge ($B = E = 0$), in which the equations take the same form as in the gauge invariant formulation, with a stress energy tensor whose spatial part is diagonal (which gives $\psi = \phi$), if we consider adiabatic perturbations with wavelengths bigger than the Hubble length, the linearized Einstein equations can be written as a conservation law for the quantity

$$\zeta = \frac{2\dot{\psi}}{3H} + \frac{5\psi}{3} \tag{2}$$

where $H = \dot{a}/a$. For a matter dominated Universe, the solution is

$$\psi(r,t) = -\frac{3}{5}\frac{t_0^{8/3} C_1(r)}{t^{5/3}} + C_2(r) \tag{3}$$

whereby $\zeta = C_2(r)$ follows. The relationship between ζ and ψ can also be written in terms of the respective Fourier modes as

$$\zeta_k = \frac{2\dot{\psi}_k}{3H} + \frac{5\psi_k}{3} \tag{4}$$

$\zeta_k = \delta\rho_k/\rho$ represents the departure from homogeneity in the density, caused by that mode of the perturbation, and is directly related to the anisotropies of the microwave background radiation, $\delta T/T$ ([18], [19]); $k^3\zeta_k$ is related to the excess mass $\delta M/M$ over a region of volume k^{-3}. From COBE measurements we know that $k^3\zeta_k \simeq 10^{-5} g(k)$, where $g(k)$ is a smooth, order one function of k [20].

B. The Tolman Universe

The Tolman metric for the motion of spherically symmetric dust [21] is

$$ds^2 = dt^2 - \frac{R'^2(r,t)}{f^2(r)}dr^2 - R^2(r,t)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{5}$$

The Einstein field equations reduce to a single equation [21]

$$\dot{R}^2(r,t) - \frac{F(r)}{R(r,t)} = f^2(r) - 1 \tag{6}$$

where $R(r,t) > 0$ and $f^2(r)$ and $F(r)$ are two arbitrary functions. $F(r)$ can be interpreted as the mass contained in a sphere, and $f^2(r) - 1$ as the mechanical energy of a shell, both of radius r.

If $f^2(r) - 1$, the Tolman equation can be solved explicitly to yield

$$R(r,t) = \left[\frac{9}{4}F(r)\right]^{1/3} (t - t_0)^{2/3} \tag{7}$$

Otherwise, we do not have an explicit solution, but the relationship of R to t can still be given in terms of a parameter η as follows:

$f^2 > 1$

$$R(r,\eta) = \frac{F(r)}{2[f^2(r) - 1]}(\cosh\eta - 1) \tag{8}$$

$$t(r,\eta) = \frac{F(r)}{2[f^2(r) - 1]^{3/2}}(\sinh\eta - \eta) \tag{9}$$

$f^2 < 1$

$$R(r,\eta) = \frac{F(r)}{2[f^2(r) - 1]}(1 - \cos\eta) \tag{10}$$

$$t(r,\eta) = \frac{F(r)}{2[f^2(r) - 1]^{3/2}}(\eta - \sin\eta) \tag{11}$$

If $f^2 - 1 \sim kr^2$ and $F(r) \sim r^3$, the Tolman solution reduces to a dust dominated FRW Universe. In this limit, η becomes the well known "conformal" time. Observe that, in this case $f^2(r)$ equal to, larger, or less than 1, corresponds to flat, open or closed spatial sections respectively. The Tolman solution allows for a third arbitrary function of r, namely, an additive constant in the right hand sides of eqs. (9) and (11). However, we shall make the simplifying assumption of disregarding this extra freedom, which can be shown to be of little relevance to the results below.

C. Matching FRW with Perturbations to Tolman

Let us now rewrite the perturbed FRW Universe of subsection **A** in the shape of a Tolman metric. To achieve this, we perform an infinitesimal change of coordinates $t_{Tolman} = t_{FRW} + \mathcal{T}(r,t)$, $r_{Tolman} = r_{FRW} + \mathcal{R}(r,t)$, and match $g_{\mu\nu}^{Tolman} = g_{\mu\nu}^{FRW}$ and $K_{ij}^{Tolman} = K_{ij}^{FRW}$ (K_{ij} the extrinsic curvature of a spacelike surface) at the surface of matter - radiation equilibrium. Starting from the longitudinal gauge we get

$$\mathcal{T}(r,t) = -\frac{9}{10}\frac{t_0^{8/3}C_1(r)}{t^{2/3}} - C_2(r)t \tag{12}$$

$$\mathcal{R}(r,t) = \frac{9}{10}\frac{t_0^4 C_1'(r)}{t} - \frac{3}{2}t_0^{4/3}C_2'(r)t^{2/3} \tag{13}$$

$$f^2(r) = 1 - \frac{10}{3}rC_2'(r) \tag{14}$$

$$F(r) = \frac{4r^3}{9t_0^2}[1 - 5C_2(r)] \tag{15}$$

We obtain the same expressions for $f^2(r)$ and $F(r)$ if we start from the synchronous gauge, $\mathcal{T}$ and $\mathcal{R}$ being much simpler. We also note that the expressions for $f^2(r)$ and $F(r)$ depend only on $C_2(r)$, the growing mode [22]. In general the r that appears in the FRW expressions is a generic length scale while the one in the Tolman expressions is the coordinate distance from the center of the Universe. In our matching procedure, the FRW coordinate becomes a radial coordinate. This is not contradictory in view of the fact that FRW Universes can be considered a special case of the Tolman solution.

The relevant growth of perturbations starts at t_0, the beginning of the matter dominated epoch of the Universe. Modes that reentered the horizon before this time at most grow logarithmically, and short wavelength modes are suppressed through free streaming and Landau damping [19]. Therefore at t_0 we can write the perturbation as a superposition of all modes with wave number smaller than a certain cut-off k_0, corresponding to the commoving wavelength of the smallest structure that could be formed. We then have

$$C_2(r) = \int_0^{k_0} k^2 dk \zeta_k \frac{\sin kr}{kr} \tag{16}$$

where we took the Fourier expansion in spherically symmetric plane waves due to the symmetry of the Universe. ζ_k is the spectrum of primeval fluctuations, and is to be specified on a $t = const.$ surface.

Let us write

$$\zeta_k = \beta(k_0)\xi(u, k_0) \tag{17}$$

where $u = k/k_0$ and ξ is an order one function. Then $C_2(r)$ will take the form

$$C_2(r) \sim \gamma(k_0)\xi(x) \tag{18}$$

where $x = k_0 r$ and

$$\xi(x) = \frac{1}{x}\int_0^1 du\, u^2 \xi(u) \frac{\sin xu}{u} \tag{19}$$

The form of $\gamma(k_0)$ depends on the functional form for $\xi(k)$, which for the moment we leave completely general. In Section **VI** we will find concrete form for $\gamma(k_0)$ by choosing a particular spectrum for the perturbations.

¿From now on we will employ x, rather than r itself, as radial coordinate. Of course, this coordinate change does not affect the form of the Tolman metric. The expressions for $F(r)$ and $f^2(r)$ become

$$F(x) = \frac{4x^3}{9t_0^2 k_0^3}[1 - 5\gamma\xi(x)] \tag{20}$$

$$f^2(x) = 1 - \frac{10}{3}\gamma x \xi'(x) \tag{21}$$

III. COSMOLOGICAL OBSERVATIONS IN A TOLMAN UNIVERSE

Having at our disposal an exact solution of Einstein equations, we should in principle be able to follow the nonlinear evolution of structures in the Universe in all detail. Our observations, on the other hand, are restricted to that part of the Universe that we can "see", that is, to the past directed light cone with vertex at Earth and now. Therefore in order to make contact with directly observable magnitudes, we must first discuss how these quantities are related to the parameters in the Tolman metric, and also how they are manifest to us as we look back to the decoupling era from our present location in space time.

Of the possible observations of cosmological relevance, we have chosen to discuss in detail the dependence of luminosity distances and number counts with redshift, and the anisotropy in the temperature of the cosmic microwave background. In this section we shall introduce these concepts, discuss the shape of the past directed null geodesics in the Tolman Universe, and derive the main formulae to reconstruct the predictions of these models concerning redshift surveys and the Hubble law.

A. Luminosity distances

Let us begin our discussion introducing a notion of distance to a cosmological object which can be inferred from Earth - based observations. Let $d\Omega_o$ be the solid angle subtended by a bundle of null geodesics diverging from the observer, and let dS_o be the cross - sectional area of this bundle at some point. Then the observer area distance D_o of this point from the observer is defined by [23]

$$dS_o = D_o^2 d\Omega_o \tag{22}$$

Thus we can find D_o if we can measure the solid angle subtended by some object whose cross - sectional area can be found from astrophysical considerations. D_o is the same as the corrected luminosity distance [23] and also the same as the angular diameter distance [24]. For a Friedmann - Robertson - Walker Universe the expression for D_o is

$$D_o = a(t_s)\, x \tag{23}$$

where and t_s indicates time of source emission. For the Tolman Universe we have

$$D_o = R(x, t_s) \tag{24}$$

The luminosity distance d_ℓ is defined as

$$d_\ell = \sqrt{\frac{F_s}{F}} \tag{25}$$

where F_s is the flux of the source measured in its neighborhood and F is the observed flux. The corrected luminosity distance d is defined as

$$d = \frac{d_\ell}{(1+z)^2} \tag{26}$$

where z is the redshift of the source measured by the observer. By the reciprocity theorem [23], we have $d = D_o$, and the luminosity distance d_ℓ for the Tolman Universe becomes

$$d_\ell = R(x, t)(1+z)^2 \tag{27}$$

B. Number counts

One of the cosmological quantities used to study the large scale structures in the Universe is the number count of a certain structure (galaxy or cluster) as a function of the redshift z. The starting point to calculate this quantity is the number of sources in a section of a bundle of past null geodesics [23], which for a Tolman Universe reads [10]

$$dN = 4\pi\nu \frac{R^/ R^2}{f} dx \tag{28}$$

ν being the number density of structures [21]

$$\nu = \frac{F^/(x)}{16\pi M_s R^/ R^2} \tag{29}$$

and where M_s is the rest mass of the structure. The number of sources which lie at radial coordinate distances less than x as seen by an observer at $x = 0$ is [10]

$$N(x) = \frac{1}{4M_s} \int_{LC} dx \frac{F^{/}(x)}{f(x)} \tag{30}$$

where the integration is made along the light-cone, LC, parametrized by x. For an almost spatially flat Universe we can consider that this number (omitting the mass-dependent prefactor) is

$$N(x) = F(x) \tag{31}$$

This quantity ought to be evaluated along a past null geodesic.

C. Redshift

The redshift of a source as measured by an observer is defined in terms of frequencies by

$$1 + z = \frac{\nu_{emitter}}{\nu_{observer}} \tag{32}$$

where $\nu_{emitter}$ is the intrinsic frequency of the source and $\nu_{observer}$ is the value of that frequency detected by the observer. If λ is an (affine) parameter along the null geodesic between source and observer and t is the cosmological time, (32) can be written as

$$1 + z = \frac{(dt/d\lambda)|_x}{(dt/d\lambda)|_{x=0}} \tag{33}$$

where x denotes the position of the source and where we take the observer located at $x = 0$. We shall discuss further this formula in next subsection.

D. Anisotropies in the CMBR temperature

In the Tolman Universe, when we move the observer away from the center of symmetry, the observations that she/he performs will be anisotropic. The physical system on which the most sensitive analysis of anisotropies are being performed is the CMBR. In the case of the Tolman Universe there are two sources of anisotropies: one related to the fluctuations in the density at the surface of last scattering, $\triangle T/T \simeq (1/3)[\Delta\delta/\bar{\delta}]_{dec}$, and the other related to the different paths of the photons from the last scattering surface to the non-centered observer, $\triangle T/T \simeq [z(\theta) - \bar{z}]_{dec}$. We will calculate both anisotropies for our model and compare one to the other.

IV. RADIAL GEODESICS IN THE TOLMAN UNIVERSE

Having derived the generic expressions for luminosity distances and number counts in terms of Tolman coordinates and the F and f functions, the only remaining step is to find the evolution of these coordinates as we travel towards the past along a null geodesic. To this end, it is convenient to think of a null geodesic as the limiting case of a time like geodesic. Time like geodesics have a natural parametrization in terms of proper time or some other affine parameter, and the form of the geodesic can be found by integrating a Hamiltonian system. We obtain a null geodesic by further imposing the constraint that the Hamiltonian be null; the affine parametrization of the time like geodesics then induces a well defined parametrization on the null geodesic.

We can write the action for radial geodesics [25] in Tolman Universe as

$$S = \frac{1}{2} \int d\lambda \left\{ - \left(\frac{dt}{d\lambda}\right)^2 + \frac{R'^2(x,t)}{f^2(x)} \left(\frac{dx}{d\lambda}\right)^2 + R^2(x,t) \left[\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{d\lambda}\right)^2 \right] \right\} \tag{34}$$

where λ is an affine parameter for timelike and spacelike geodesics. The canonically conjugated momenta and the Hamiltonian are

$$P_t = -\frac{dt}{d\lambda} \tag{35}$$

$$P_r = \frac{R'^2(x,t)}{f^2(x)}\frac{dx}{d\lambda} \tag{36}$$

$$P_\theta = R^2(x,t)\frac{d\theta}{d\lambda} \tag{37}$$

$$P_\varphi = R^2(x,t)\sin^2\theta\frac{d\varphi}{d\lambda} \tag{38}$$

$$H = -P_t^2 + \frac{f^2(x)}{R'^2(x,t)}P_r^2 + \frac{1}{R^2(x,t)}P_\theta^2 + \frac{1}{R^2(x,t)\sin^2\theta}P_\varphi^2 \tag{39}$$

For $H \to 0$ we have a null geodesic , and in this case the redshift is given by (cfr. eq. (33))

$$1 + z = \frac{(dt/d\lambda)|_x}{(dt/d\lambda)|_{x=0}} = \frac{P_t|_x}{P_t|_{x=0}} \tag{40}$$

We choose the parameter λ so that $P_t|_{x=0} = 1$, then $P_t|_x$ is directly the redshift at that location.

In order to proceed with the integration of these equations, we should know the expression for R as a function of t and x, which entails solving the parametric equations (8) and (9) or (10) and 11), with known F and f. Obviously this cannot be done in closed form unless in the trivial case $f \equiv 1$. Nevertheless we can solve those equations perturbatively, developing the hyperbolic (or trigonometric) functions around the origin ($t \to 0$, $x \simeq 0$), the corresponding expressions for R, R^{-2} and R'^{-2} can be read in Appendix **A**.

Let us consider the evolution of the perturbations in a neighborhood of the origin. We have to develop the functions of x around $x \simeq 0$. For the function F we approximate it by

$$F(x) \sim \frac{4x^3}{9t_0^2k_0^3} \tag{41}$$

since $\gamma \sim O(10^{-5})$. From eqs. (9) and (14) we obtain

$$\sinh\eta - \eta \simeq 2t\frac{[10\gamma x\xi'(x)/3]^{3/2}}{4x^3/9t_0^2k_0^3} \tag{42}$$

(of course the same argument applies in the trigonometric case) To achieve a non null value of η at the origin we have to require that $x\xi'(x)$ goes to 0 like ax^2, where a is some constant of order one. In this case eq. (42) yields

$$\eta_{today} = 3(k_0^3t_0^2t_{today})^{1/3}\sqrt{\frac{10\gamma a}{3}} \tag{43}$$

Since η decreases as we move down the geodesic, we find that throughout we remain in a neighborhood of $\eta = 0$. We are led then to an approximation scheme whereby, in the $m - th$ order approximation, we replace the hyperbolic (or trigonometric) functions in the parametric equations by the first nontrivial m terms in their Taylor expansion around the origin. For example, at leading order we approximate $\sinh\eta - \eta \sim \eta^3/6$; at next to leading order we retain also the η^5 term, and so on. In other words, we are neglecting terms of order $\eta^{2m+2}/(2m+2)!$ against those of order $\eta^{2m}/(2m)!$. This approximation is seen to be valid if $\eta^2 \ll (2m+1)(2m+2)$. η_{today} is a measure of the non linear evolution of the perturbation: higher η_{today} means higher non linearity and therefore more orders are to be considered in the development of the hyperbolic (trigonometric) expressions of t and R. ¿From eq. (43) we see that increasing η_{today} means increasing k_0, i.e. considering a perturbation characterized by a smaller wavelength and that consequently is in a more non linear stage of evolution today.

A. Leading Order Approximation

For radial geodesics the Hamiltonian reads

$$H = -P_t^2 + \frac{f^2(x)}{R'^2(x,t)} P_x^2 \tag{44}$$

Starting from eqs. (8) and (9) and taking the lowest order in the development of the hyperbolic functions ($\eta \ll 1$) we have

$$\eta^3 \simeq \frac{12\left[f^2(x)-1\right]^{3/2} t}{F(x)} \tag{45}$$

and therefore

$$R(x,t) \simeq \frac{12^{2/3} F^{1/3}(x)\, t^{2/3}}{4} \tag{46}$$

The Hamilton-Jacobi equation for the characteristic function is

$$-\left(\frac{\partial W}{\partial t}\right)^2 + \frac{1}{v_0^2(x) t^{4/3}} \left(\frac{\partial W}{\partial x}\right)^2 = -E \tag{47}$$

with

$$v_0(x) = \frac{F'(x)}{12^{1/3} F^{2/3}(x) f(x)}$$

and where we used as the principal function $S = W + \lambda E$. We separate variables by writing $W = T(t,p,E) + pL(x)$, with

$$\frac{dL}{dx} = v_0(x) \tag{48}$$

$$\frac{dT}{dt} = \sqrt{E + \frac{p^2}{t^{4/3}}} \tag{49}$$

As we are interested only in a neighborhood of $E = 0$, we can expand eq. (49)

$$\frac{dT}{dt} \simeq \frac{p}{t^{2/3}} + \frac{1}{2} E \frac{t^{2/3}}{p} - \frac{1}{8} E^2 \frac{t^2}{p^3} + \cdots \tag{50}$$

which can be integrated to give

$$T \simeq 3pt^{1/3} + \frac{3}{10} E \frac{t^{5/3}}{p} - \frac{1}{24} E^2 \frac{t^3}{p^3} + \cdots \tag{51}$$

If we define new canonical variables τ, ρ conjugated to E and p respectively, we have from the characteristic function W that

$$\tau = \frac{\partial W}{\partial E} = \frac{3}{10} \frac{t^{5/3}}{p} - \frac{1}{12} E \frac{t^3}{p^3} + \cdots \tag{52}$$

$$\rho = \frac{\partial W}{\partial p} = L(x) + 3t^{1/3} - \frac{3}{10} E \frac{t^{5/3}}{p^2} + \frac{1}{8} E^2 \frac{t^3}{p^4} - \cdots \tag{53}$$

from where

$$L(x) = \rho - 3t^{1/3} + \frac{3}{10}E\frac{t^{5/3}}{p^2} - \frac{1}{8}E^2\frac{t^3}{p^4} + \cdots \tag{54}$$

We can write the old momentum P_t in terms of the new momenta as

$$P_t = \frac{\partial W}{\partial t} = \sqrt{E + \frac{p^2}{t^{4/3}}} = \frac{p}{t^{2/3}} + \frac{1}{2}E\frac{t^{2/3}}{p} - \frac{1}{8}E^2\frac{t^2}{p^3} + \cdots \tag{55}$$

The new Hamiltonian is $H = -E$. The equations of motion are

$$\begin{aligned} \dot{\tau} &= \partial H/\partial E = -1 \\ \dot{E} &= \dot{p} = \dot{\rho} = 0 \end{aligned} \tag{56}$$

where a dot means $d/d\lambda$. As $p = const.$ we can choose it as

$$p = t_{today}^{2/3}$$

so that

$$P_t|_{E=0} = (1+z) = \left(\frac{t_{today}}{t}\right)^{2/3} \tag{57}$$

therefore when $t = t_{today}$ we have $z = 0$. We read the spatial coordinate from eq. (53) and as also $\rho = const.$ we take it as

$$\rho = 3t_{today}^{1/3}$$

therefore from eq. (53) we have

$$L(x)|_{E=0} = 3t_{today}^{1/3} - 3t^{1/3}$$

We can use this last equation to find an expression for $F(x)$. From Eq.(48) and the definition of $v_0(x)$ we have

$$L(x) = \frac{12^{2/3}}{4}\left\{\frac{F^{1/3}(x)}{f(x)} + \int dx \frac{F^{1/3}(x)f'(x)}{f^2(x)}\right\} \tag{58}$$

As the second term is $O(\gamma)$ we can neglect it. So for $f(x) \sim 1$ we have

$$F^{1/3}(x) = \frac{4}{12^{2/3}}L(x) = (12t_{today})^{1/3}\left[1 - \frac{t^{1/3}}{t_{today}^{1/3}}\right] \tag{59}$$

Finally, recalling Eq. (41) the equation for the geodesic becomes

$$x = x_0\left[1 - \frac{t^{1/3}}{t_{today}^{1/3}}\right] \tag{60}$$

where $x_0 = 3(k_0^3 t_0^2 t_{today})^{1/3} = (10\gamma a/3)^{-1/2}\eta_{today}$.

According to eqs. (59), (46), (57) and (27) the luminosity distance reads

$$d_\ell = 3t_{today}\left[(1+z) - \sqrt{1+z}\right] \tag{61}$$

where we see that for small z, $d_\ell \simeq H_0^{-1}z$, $H_0 = 2/3t_{today}$ as expected for a flat, matter dominated FRW Universe.

B. Next to Leading Order Approximation

Now the Hamilton-Jacobi equation reads

$$-\left(\frac{\partial W}{\partial t}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2 \frac{1}{v_0^2(x)t^{4/3}}\left[1 - \frac{1}{10}v_1(x)t^{2/3}\right] = -E \tag{62}$$

where

$$v_1(x) = \frac{12^{2/3}}{[F^{1/3}(x)]'}\left(\frac{f^2(x)-1}{F^{1/3}(x)}\right)' = -\frac{\eta_{today}^2}{at_{today}^{2/3}}\xi''(x) \tag{63}$$

E and p are no longer constants of motion because in terms of them

$$H = -E - p^2\frac{v_1(x)}{10t^{2/3}} \tag{64}$$

We therefore perform a further canonical transformation to variables E_1, p_1, ρ_1, τ_1 so that $H = -E_1$. The generating function is now

$$W_1 = E_1\tau + p_1\rho - p_1^2 G_1(E_1, p_1, \rho, \tau) \tag{65}$$

The old momentum E becomes

$$E = \frac{\partial W_1}{\partial \tau} = E_1 - p_1^2\frac{\partial G_1}{\partial \tau} \tag{66}$$

so to obtain $H = -E_1$ we demand

$$\frac{\partial G_1}{\partial \tau} = \frac{v_1(x)}{10t^{2/3}} \tag{67}$$

We can write formally

$$G_1 = \frac{1}{10}\int_{\tau_{today}}^{\tau} d\tau' \frac{v_1[x(E_1, p_1, \rho, \tau')]}{t^{2/3}(E_1, p_1, \tau')} \tag{68}$$

Under the integration sign we may use the zeroth order relationships among the different variables, namely $E_1 \sim E$, $p_1 \sim p$, and

$$\tau = \frac{dT}{dE} = \frac{d}{dE}\left(\int_{t_{today}}^{t} dt'\sqrt{E + \frac{p^2}{t^{4/3}}}\right) \to d\tau = \frac{1}{2p}\frac{t^{2/3}}{\sqrt{1+\frac{Et^{4/3}}{p^2}}}dt \tag{69}$$

so

$$G_1 = \frac{1}{20p_1}\int_{t_{today}}^{t} \frac{dt'}{\sqrt{1+\frac{E_1 t^{4/3}}{p_1^2}}} v_1[x(E_1, p_1, \rho, t')] \tag{70}$$

where $t = t(\tau, E_1, p_1)$. Let us now compute the corrections to the several quantities introduced so far.

1. Corrections to E

The value of E valid to this order follows immediately from eqs. (66) and (67) plus the Hamiltonian constraint $E_1 = 0$, leading to

$$E = -\frac{p_1^2}{10}\frac{v_1(x)}{t^{2/3}}$$

where $t = t(\tau_1, E_1 = 0, p_1)$, $x = x(\tau_1, \rho_1, E_1 = 0, p_1)$.

2. Corrections to p

The value of the old momentum p follows from

$$p = \frac{\partial W_1}{\partial \rho} = p_1 - p_1^2 \frac{\partial G_1}{\partial \rho}$$

Eq. (70) leads to

$$\frac{\partial G_1}{\partial \rho} = \frac{1}{20 p_1} \int\limits_{t_{today}}^{t} \frac{dt'}{\sqrt{1 + \frac{E_1 t^{4/3}}{p_1^2}}} v_1'(x) \frac{\partial x}{\partial \rho}|_{E_1, p_1, t'} \tag{71}$$

where $v_1'(x) = dv_1(x)/dx$ and where the integration is to be performed along a past null geodesic. As $v_1(x)$ is of first order, $\partial x/\partial \rho$ may be evaluated from the zeroth order, giving $\partial x/\partial \rho = x_0/3t_{today}^{1/3}$. Therefore, setting $E_1 = 0$ we get

$$\frac{\partial G_1}{\partial \rho}|_{E_1=0} = \frac{1}{20 p_1} \frac{x_0}{3t_{today}^{1/3}} \int\limits_{t_{today}}^{t} dt' v_1'(x) \tag{72}$$

and

$$p = p_1 \left\{ 1 - \frac{1}{20} \frac{x_0}{3t_{today}^{1/3}} \int\limits_{t_{today}}^{t} dt' v_1'(x) \right\} \tag{73}$$

3. Corrections to ρ and τ

The new coordinates are

$$\rho_1 = \frac{\partial W_1}{\partial p_1} = \rho - 2p_1 G_1 - p_1^2 \frac{\partial G_1}{\partial p_1} \tag{74}$$

$$\tau_1 = \frac{\partial W_1}{\partial E_1} = \tau - p_1^2 \frac{\partial G_1}{\partial E_1} \tag{75}$$

For ρ we have from (70) that at $E_1 = 0$

$$\frac{\partial G_1}{\partial p_1} = -\frac{G_1}{p_1} + \frac{1}{20 p_1} v_1(x) \frac{\partial t}{\partial p_1} - \frac{1}{20 p_1} \int\limits_{t_{today}}^{t} dt' v_1'(x) \frac{\partial x}{\partial p_1}|_{E,\rho,t} \tag{76}$$

¿From the zeroth order we have $\rho = 3t^{1/3} + L(x)$, and consequently $\partial x/\partial p = 0$ and from $t = (10p\tau/3)^{3/5}$ we get $\partial t/\partial p|_\tau = 3t/5p_1$. Therefore

$$\rho = \rho_1 + \frac{3t}{100} v_1(x) + \frac{1}{20} \int_{t_{today}}^{t} dt' v_1(x) \tag{77}$$

Finally for τ we have (cf. Eq. (75))

$$\tau = \tau_1 + \frac{1}{12p_1} v_1(x)\, t^{7/3} + \frac{x_0}{20 t_{today}^{1/3} p_1} \int_{t_{today}}^{t} dt\, t^{5/3}\, v_1'(x) - \frac{1}{4p_1} \int_{t_{today}}^{t} dt\, t^{5/3}\, v_1(x) \tag{78}$$

where we have used the zeroth order results $\partial t/\partial E_1|_{E=0} = t^{7/3}/6p^2$ and $\partial x/\partial E_1|_{E=0} = x_0 t^{5/3}/10 t_{today}^{1/3} p^2$.

4. Redshift, Number Count, Density Contrast and Luminosity Distance

Having computed the old canonical coordinates in terms of the new ones, it is only a matter of substituting them into eqs. (54) and (55) to obtain the formulae for the number counts (through the auxiliary function $L(x)$, recall Eq. (59)) and redshifts. For $L(x)$ we get

$$L(x) = \rho - 3t^{1/3} + \frac{3}{10} E \frac{t^{5/3}}{p^2} = \rho_1 + \frac{1}{20} \int_0^t dt' v_1(x) - 3t^{1/3} \tag{79}$$

Imposing the boundary condition $L(0) = 0$, we get

$$L(x) = 3(t_{today}^{1/3} - t^{1/3}) + \frac{1}{20} \int_t^{t_{today}} dt' v_1(x) \tag{80}$$

Substituting the expressions for E and p in Eq. (55) for P_t, and imposing the boundary condition $P_t = 1$ at $t = t_{today}$ we get

$$P_t = \left(\frac{t_{today}}{t}\right)^{2/3} \left\{ 1 - \frac{1}{20} \left(t^{2/3} v_1(x) - t_{today}^{2/3} v_1(0) \right) - \frac{1}{20p_1} \frac{x_0}{3 t_{today}^{1/3}} \int_{t_{today}}^{t} dt' v_1'(x) \right\} \tag{81}$$

Now we must find the explicit expressions for $dt = dt(x)$. From the zeroth order we have

$$dt = -\frac{3t_{today}}{x_0} \left(1 - \frac{x}{x_0}\right)^2 dx \simeq -\frac{3t_{today}}{x_0} dx \tag{82}$$

where the last expression is due to the fact that we are interested in a neighborhood of $x = 0$. Replacing this expression and Eq. (63) in eqs. (79) and (81) we get

$$P_t = \left(\frac{t_{today}}{t}\right)^{2/3} \left\{ 1 - \frac{\eta_{today}^2}{20a} \xi''(x) \left[1 - \left(\frac{t}{t_{today}}\right)^{2/3} \right] \right\} \tag{83}$$

$$F^{1/3}(x) = (12t_{today})^{1/3} \left\{ 1 - \left(\frac{t}{t_{today}}\right)^{1/3} + \frac{\eta_{today}}{20} \sqrt{\frac{10\gamma}{3a}} \xi'(x) \right\} \tag{84}$$

and in this case the number count is given by $N(x) = F(x)$ (cfr. Eq. (30)). The density contrast is given by eq. (29) omitting the mass factor and in this case we obtain

$$\frac{\Delta\delta}{\bar{\delta}} = \frac{\eta_{today}^2}{20a}\left[\xi^{''}(x) + 2\frac{\xi^{'}(x)}{x}\right]\left(\frac{t}{t_{today}}\right)^{2/3} \tag{85}$$

where $\bar{\delta} = (12\pi t^2)^{-1}$ and where we used δ for the energy density in order to avoid confusion with notation. Finally, the luminosity distance reads

$$d_\ell = 3\frac{t_{today}^{5/3}}{t^{2/3}}\left\{1 - \frac{t^{1/3}}{t_{today}^{1/3}} + \frac{\eta_{today}}{20}\sqrt{\frac{10\gamma}{3a}}\left[\xi^{'}(x) - 2x\xi^{''}(x)\right]\left[1 - \left(\frac{t}{t_{today}}\right)^{2/3}\right]\right\} \tag{86}$$

where these last three quantities are to be evaluated using $x = x_0(1 - t^{1/3}/t_{today}^{1/3})$. These expressions are completely general, the only assumption on the function $\xi(x)$ being that $x\xi^{'}(x)$ goes to zero as x^2. Therefore, given a power spectrum ζ_k we can find the perturbed background, described by $F(x)$ and $f^2(x)$ at a given initial time t_0 and then evolve it non linearly using the Tolman equations. However, to properly speak about non linear evolution, we have to consider higher orders. The procedure to evaluate them is the same as the used for the next to leading order. In the next subsection we sketch the evaluation of the second order and give the final results.

C. Second Order Approximation

In this case the Hamilton - Jacobi equation reads

$$-\left(\frac{\partial W}{\partial t}\right)^2 + \left(\frac{\partial W}{\partial x}\right)^2 \frac{1}{v_0^2(x)t^{4/3}}\left[1 - \frac{1}{10}v_1(x)t^{2/3} + v_2(x)t^{4/3}\right] = -E \tag{87}$$

where

$$\begin{aligned} v_2(x) &= \frac{3}{400}\frac{12^{4/3}}{[F^{1/3}(x)]^{'2}}\left[\frac{f^2(x)-1}{F^{1/3}(x)}\right]^{'2} + \frac{3}{1400}\frac{12^{4/3}}{[F^{1/3}(x)]^{'}}\left[\frac{(f^2(x)-1)^2}{F}\right]^{'} \\ &= \frac{3}{400a^2}\frac{\eta^4}{t_{today}^{4/3}}\xi^{''2}(x) + \frac{3}{1400a^2}\frac{\eta^4}{t_{today}^{4/3}}\left(\frac{\xi^{'2}}{x}\right)^{'} \end{aligned}$$

If we replace in (87) the leading and next to leading order we find again that E_1 and p_1 are no longer constants of motion because in terms of them

$$H = -E_1 + p_1^2 v_2(x)$$

We perform a further canonical transformation whose generating function is again of the form

$$W_2 = E_2\tau_1 + p_2\rho_1 - p_2^2 G_2(\rho_1, \tau_1, p_2, E_2)$$

so the first order momentum E_1 becomes

$$E_1 = \frac{\partial W_2}{\partial \tau_1} = E_2 - p_2^2\frac{\partial G_2}{\partial \tau_1}$$

To obtain $H = -E_2$ we proceed as in the previous order and demand

$$\frac{\partial G_2}{\partial \tau_1} = -v_2(x)$$

The procedure to calculate the corrections to p_1, ρ_1 is the same, although longer, as in the next to leading order case. We therefore show only the final expressions for E_1, p_1, ρ_1, P_t, $F^{1/3}(x)$, $\Delta\delta/\delta$ and d_ℓ.

$$E_1 = p_2^2 v_2(x) \tag{88}$$

$$p_1 = p_2 \left\{ 1 + \frac{x_0}{6t_{today}^{1/3}} \int_{t_{today}}^{t} dt\, t^{2/3} v_2^{'}(x) \right\} \tag{89}$$

$$\rho_1 = \rho_2 - \frac{7}{10} \int_{t_{today}}^{t} dt\, t^{2/3} v_2(x) - \frac{3}{10} t^{5/3} v_2(x) \tag{90}$$

$$\begin{aligned} P_t = \left(\frac{t_{today}}{t}\right)^{2/3} \Bigg\{ & 1 - \frac{\eta^2}{20a}\xi^{''}(x)\left[1 - \frac{t^{2/3}}{t_{today}^{2/3}}\right] - \frac{3\eta^4}{2800a^2}\left(\frac{\xi^{'2}(x)}{x}\right)^{'}\left[1 - \frac{t^{4/3}}{t_{today}^{4/3}}\right] - \\ & - \frac{3\eta^4}{800a^2}\xi^{''2}(x)\left[1 - \frac{t^{4/3}}{t_{today}^{4/3}}\right] - \frac{\eta^4}{800a^2}\xi^{''2}(x)\left[\frac{t^{2/3}}{t_{today}^{2/3}} - 2\frac{t^{4/3}}{t_{today}^{4/3}}\right] - \frac{\eta^4}{800a^2}\xi^{''2}(0) \Bigg\} \end{aligned} \tag{91}$$

$$\begin{aligned} F^{1/3}(x) = (12t_{today})^{1/3} \Bigg\{ & 1 - \frac{t^{1/3}}{t_{today}^{1/3}} + \frac{\eta_{today}}{20}\sqrt{\frac{10\gamma}{3a}}\xi^{'}(x) + \\ & + \frac{21\eta_{today}^3}{4000}\sqrt{\frac{10\gamma}{3a^3}} \int_0^x dx \xi^{''2}(x) + \frac{3\eta_{today}^3}{2000}\sqrt{\frac{10\gamma}{3a^3}}\frac{\xi^{'2}(x)}{x} - \\ & - \frac{\eta_{today}^4}{2400a^2}\xi^{''2}(x)\frac{t^{5/3}}{t_{today}^{5/3}} + \frac{\eta_{today}^4}{2400a^2}\xi^{''2}(0) + \\ & + \frac{\eta_{today}^4}{1000a^2}\xi^{''2}(x)\frac{t}{t_{today}} - \frac{\eta_{today}^4}{1000a^2}\xi^{''2}(0) \Bigg\} \end{aligned} \tag{92}$$

$$\begin{aligned} \frac{\Delta\delta}{\bar{\delta}} = & \frac{\eta_{today}^2}{20a}\left[\xi^{''}(x) + 2\frac{\xi^{'}(x)}{x}\right]\left(\frac{t}{t_{today}}\right)^{2/3} - \frac{\eta_{today}^4}{700a^2}\frac{\xi^{'2}(x)}{x^2}\left(\frac{t}{t_{today}}\right)^{4/3} + \\ & + \frac{\eta_{today}^4}{140a^2}\frac{\xi^{'}(x)\xi^{''}(x)}{x}\left(\frac{t}{t_{today}}\right)^{4/3} + \frac{\eta_{today}^4}{400a^2}\frac{\xi^{'2}(x)}{x^2}\left[4\left(\frac{t}{t_{today}}\right)^{4/3} - 3\left(\frac{t}{t_{today}}\right)^{2/3}\right] + \\ & + \frac{\eta_{today}^4}{400a^2}\xi^{''2}(x)\left[\left(\frac{t}{t_{today}}\right)^{4/3} - \left(\frac{t}{t_{today}}\right)^{2/3}\right] - \frac{\eta_{today}^4}{200a^2}\frac{\xi^{'}(x)\xi^{''}(x)}{x}\left(\frac{t}{t_{today}}\right)^{2/3} \end{aligned} \tag{93}$$

$$\begin{aligned} d_\ell = 3\frac{t_{today}^{5/3}}{t^{2/3}} \Bigg\{ & \left(1 - \frac{t^{1/3}}{t_{today}^{1/3}}\right) + \frac{\eta_{today}}{20}\sqrt{\frac{10\gamma}{3a}}\left[\xi^{'}(x) - 2x\xi^{''}(x)\right]\left(1 - \frac{t^{2/3}}{t_{today}^{2/3}}\right) + \\ & + \frac{21\eta_{today}^3}{4000}\sqrt{\frac{10\gamma}{3a^3}}\int_0^x dx \xi^{''2}(x) - \frac{\eta_{today}^3}{400}\sqrt{\frac{10\gamma}{3a^3}} x\xi^{''2}(x)\left(2 - 3\frac{t^{4/3}}{t_{today}^{4/3}}\right) - \end{aligned} \tag{94}$$

$$
-\frac{\eta_{today}^3}{1400}\sqrt{\frac{10\gamma}{3a^3}}\xi^{'}(x)\xi^{''}(x)\left(13-14\frac{t^{2/3}}{t_{today}^{2/3}}+\frac{t^{4/3}}{t_{today}^{4/3}}\right)+ \tag{95}
$$

$$
+\frac{\eta_{today}^3}{2800}\sqrt{\frac{10\gamma}{3a^3}}\frac{\xi^{'2}(x)}{x}\left(6+7\frac{t^{2/3}}{t_{today}^{2/3}}-9\frac{t^{4/3}}{t_{today}^{4/3}}\right)+\frac{3\eta_{today}^3}{2000}\sqrt{\frac{10\gamma}{3a^3}}\frac{\xi^{'2}(x)}{x}-
$$

$$
\left.-\frac{\eta_{today}^4}{2400a^2}\xi^{''2}(x)\frac{t^{5/3}}{t_{today}^{5/3}}+\frac{\eta_{today}^4}{2400a^2}\xi^{''2}(0)+\frac{\eta_{today}^4}{1000a^2}\xi^{''2}(x)\frac{t}{t_{today}}-\frac{\eta_{today}^4}{1000a^2}\xi^{''2}(0)\right\}
$$

and here again, $N(x) = F(x)$. Observe that the new corrections are obviously non linear in the perturbations.

The derivation of these formulae is valid irrespective of the details of the primordial spectrum $\xi(x)$, and so this method may be used to work out in a simple fashion the predictions of several scenarios of primordial fluctuation generation.

V. THE VIEW FROM EARTH

Due to the spherical symmetry of the Tolman Universe, it is not possible to study anisotropic effects using radial geodesics, i.e. an observer located in the center of the Universe, will detect no anisotropies. To study the presence of anisotropies in a physical system using the Tolman solution to Einstein equations we have to move the observer away from the center of symmetry and solve for it the null geodesics equation. In this section we solve the mentioned equation to both leading and next to leading order, and find expressions to analyze the anisotropies in the CMBR due to the different redshifts of photons, travelling from the decoupling surface to the present.

Due to the spherical symmetry of the Tolman Universe we do not loose generality if we locate the observer at $\theta = 0$. In this case the Hamilton Jacobi equation for the characteristic function reads

$$
-\left(\frac{\partial W}{\partial t}\right)^2+\frac{f^2(x)}{R'^2(x,t)}\left(\frac{\partial W}{\partial x}\right)^2+\frac{1}{R^2(x,t)}\left(\frac{\partial W}{\partial \theta}\right)^2=-E \tag{96}
$$

where we used as the principal function $S = W + \lambda E$. We separate variables by writing $W = T(t, p, E) + L(x, p, \alpha) + M(\alpha, \theta)$.

A. Leading Order Approximation

We rewrite Eq. (96) as

$$
-\left(\frac{\partial W}{\partial t}\right)^2+\frac{1}{v_0^2 t^{4/3}}\left(\frac{\partial W}{\partial x}\right)^2+\frac{1}{v_0^2 x^2 t^{4/3}}\left(\frac{\partial W}{\partial \theta}\right)^2=-E \tag{97}
$$

with $v_0^2 = 9t_{today}^{2/3}/x_0^2$ and where we have used (41) to evaluate $R(x,t)$ and $R^{'}(x,t)$. In this case we have

$$
\frac{dM}{d\theta}=v_0\alpha \rightarrow M=v_0\alpha\theta \tag{98}
$$

$$
\frac{dL}{dx}=P_r=v_0\sqrt{p^2-\frac{\alpha^2}{x^2}} \rightarrow L=v_0\left[\sqrt{p^2x^2-\alpha^2}-\alpha\arccos\left(\frac{\alpha}{px}\right)\right] \tag{99}
$$

$$
\frac{dT}{dt}=\sqrt{E+\frac{p^2}{t^{4/3}}} \tag{100}
$$

As we are interested in a neighborhood of $E = 0$, we can expand

$$\frac{dT}{dt} \simeq \frac{p}{t^{2/3}} + \frac{1}{2}E\frac{t^{2/3}}{p} - \frac{1}{8}E^2\frac{t^2}{p^3} + \cdots \tag{101}$$

which can be integrated to give

$$T \simeq 3pt^{1/3} + \frac{3}{10}E\frac{t^{5/3}}{p} - \frac{1}{24}E^2\frac{t^3}{p^3} + \cdots \tag{102}$$

If we define new canonical variables τ, $\tilde{\theta}$ and ρ conjugated to E, α and p respectively, we have from the characteristic function W that

$$\tau = \frac{\partial W}{\partial E} = \frac{3}{10}\frac{t^{5/3}}{p} - \frac{1}{12}E\frac{t^3}{p^3} + \cdots \tag{103}$$

$$\rho = \frac{\partial W}{\partial p} = \frac{v_0}{p}\sqrt{p^2x^2 - \alpha^2} + 3t^{1/3} - \frac{3}{10}E\frac{t^{5/3}}{p^2} + \cdots \tag{104}$$

$$\chi = \frac{\partial W}{\partial \alpha} = v_0\left[\theta - \arccos\left(\frac{\alpha}{px}\right)\right] \tag{105}$$

¿From (104) and recalling that $F^{1/3}(x) \simeq (12t_{today})^{1/3}x/x_0$ we get

$$\frac{x_0}{(12t_{today})^{1/3}}F^{1/3}(x) = \frac{1}{p}\sqrt{\frac{p^2}{\nu_0^2}\left(\rho - 3t^{1/3} + \frac{3}{10}E\frac{t^{5/3}}{p^2} - \cdots\right)^2 + \alpha^2} \tag{106}$$

from where we will read the expression for $F^{1/3}(x)$.

We can write the old momentum P_t in terms of the new variables as

$$P_t = \frac{\partial W}{\partial t} = \sqrt{E + \frac{p^2}{t^{4/3}}} \simeq \frac{p}{t^{2/3}} + \frac{1}{2}E\frac{t^{2/3}}{p} - \frac{1}{8}E^2\frac{t^2}{p^3} + \cdots \tag{107}$$

The new Hamiltonian is $H = -E$. The equations of motion are

$$\begin{aligned} \dot{\tau} &= \partial H/\partial E = -1 \\ \dot{E} &= \dot{p} = \dot{\rho} = \dot{\alpha} = \dot{\chi} = 0 \end{aligned} \tag{108}$$

where a dot means $d/d\lambda$. As $p = const$ we can choose it as

$$p = t_{today}^{2/3} \tag{109}$$

so that

$$P_t \mid_{E=0} = (1+z) = \left(\frac{t_{today}}{t}\right)^{2/3} \tag{110}$$

therefore when $t = t_{today}$ we have $z = 0$. In order to determine α we need to specify the position of the observer, x_T and the angle of observation, φ. We calculate the observer's position by adjusting the velocity of the Earth to the value determined by the dipolar contribution to the background radiation anisotropy, i.e.

$$\dot{R} = 2t_{today}^{1/3}\frac{x}{x_0}t^{-1/3} \mid_{t=today} = 1.666 \times 10^{-3} \rightarrow x_T = 0.833 \times 10^{-4}x_0 \tag{111}$$

The angle subtended by a null ray that reaches x_T at t_{today} is defined through (see Fig. **1**)

$$v_0 \cos(\pi - \varphi) = \frac{P_r}{p} \tag{112}$$

from where we obtain

$$\alpha = p x_T \sin\varphi \tag{113}$$

As ρ is a constant of the motion, we fix its value by replacing in eq. (104) $x = x_T$, $t = t_{today}$, i.e. we choose

$$\rho = 3t_{today}\left[1 + \frac{x}{x_0}\cos\varphi\right] \tag{114}$$

Therefore the equation for the geodesics reads

$$x = x_0\sqrt{\left(1 - \frac{t^{1/3}}{t_{today}^{1/3}}\right)^2 + 2\left(1 - \frac{t^{1/3}}{t_{today}^{1/3}}\right)\frac{x_T}{x_0}\cos\varphi + \frac{x_T^2}{x_0^2}} \tag{115}$$

and the redshift is given by

$$1 + z = \frac{t_{today}^2}{t^{2/3}} \tag{116}$$

B. Next to Leading Order Approximation

In this case the Hamilton-Jacobi equation for the characteristic function reads

$$-\left(\frac{\partial W}{\partial t}\right)^2 + \frac{1}{v_0^2 t^{4/3}}[1 - v_1(x)t^{2/3}]\left(\frac{\partial W}{\partial x}\right)^2 + \frac{1}{v_0^2 t^{4/3}}[1 - w_1(x)t^{2/3}]\left(\frac{\partial W}{\partial \theta}\right)^2 = -E \tag{117}$$

where

$$\begin{aligned} v_1 &= \frac{1}{10}\frac{x_0^2}{t_{today}^{2/3}}\left[\frac{f^2(x) - 1}{x}\right]' = -\frac{\eta^2}{10 a t_{today}^{2/3}}\xi''(x) \\ w_1 &= \frac{1}{10}\frac{x_0^2}{t_{today}^{2/3}}\left[\frac{f^2(x) - 1}{x^2}\right] = -\frac{\eta^2}{10 a t_{today}^{2/3}}\frac{\xi'(x)}{x} \end{aligned} \tag{118}$$

(cfr. Appendix **A**)

Now, E, p and α are no longer constants of motion because in terms of them

$$H = -E - p^2\frac{v_1(x)}{t^{2/3}} - \frac{\alpha^2}{x^2 t^{2/3}}\left[w_1(x) - v_1(x)\right] \tag{119}$$

We therefore perform further canonical transformation to variables E_1, p_1, α_1, ρ_1, χ_1, τ_1 so that $H = -E_1$. The generating function is now

$$W_1 = E_1\tau + p_1\rho + \alpha_1\chi - p_1^2\mathcal{G}_1(\rho, \chi, \tau, p_1, \alpha_1, E_1) - \alpha_1^2\mathcal{F}_1(\rho, \chi, \tau, p_1, \alpha_1, E_1) \tag{120}$$

The old momentum E becomes

$$E = \frac{\partial W_1}{\partial \tau} = E_1 - p_1^2\frac{\partial \mathcal{G}_1}{\partial \tau} - \alpha_1^2\frac{\partial \mathcal{F}_1}{\partial \tau} \tag{121}$$

so to obtain $H = -E_1$ we demand

$$\frac{\partial \mathcal{G}_1}{\partial \tau} = \frac{v_1(x)}{t^{2/3}} \tag{122}$$

$$\frac{\partial \mathcal{F}_1}{\partial \tau} = \frac{1}{x^2 t^{2/3}} \left[w_1(x) - v_1(x) \right] \tag{123}$$

We can write formally

$$\mathcal{G}_1 = \int\limits_{\tau_{today}}^{\tau} d\tau' \frac{v_1[x(\rho, \chi, \tau', p_1, \alpha_1, E_1)]}{t^{2/3}(E_1, p_1, \tau')} \tag{124}$$

$$\mathcal{F}_1 = \int\limits_{\tau_{today}}^{\tau} d\tau' \frac{w_1[x(\rho, \chi, \tau', p_1, \alpha_1, E_1)] - v_1[x(\rho, \chi, \tau', p_1, \alpha_1, E_1)]}{x^2(\rho, \chi, \tau', p_1, \alpha_1, E_1)\; t^{2/3}(E_1, p_1, \tau')} \tag{125}$$

In terms of the zeroth order variables we can write (holding fixed E_1, p_1)

$$\tau = \frac{dT}{dE} = \frac{d}{dE} \left[\int\limits_{t_{today}}^{t} dt \frac{p}{t^{2/3}} \sqrt{1 + E\frac{t^{4/3}}{p^2}} \right] \rightarrow d\tau = \frac{1}{2p} \frac{t^{2/3} dt}{\sqrt{1 + E\frac{t^{4/3}}{p^2}}} \tag{126}$$

so

$$\mathcal{G}_1 = \frac{1}{2p_1} \int\limits_{\tau_{today}}^{\tau} \frac{dt}{\sqrt{1 + E\frac{t^{4/3}}{p^2}}} v_1[x(\rho, \chi, \tau, p_1, \alpha_1, E_1)] \tag{127}$$

$$\mathcal{F}_1 = \frac{1}{2p_1} \int\limits_{\tau_{today}}^{\tau} \frac{dt}{\sqrt{1 + E\frac{t^{4/3}}{p^2}}} \frac{w_1[x(\rho, \chi, \tau, p_1, \alpha_1, E_1)] - v_1[x(\rho, \chi, \tau, p_1, \alpha_1, E_1)]}{x^2(\rho, \chi, \tau, p_1, \alpha_1, E_1)} \tag{128}$$

where $t = t(E_1, p_1, \tau)$. Let us now compute the corrections to the several quantities introduced so far.

1. Corrections to E

The value of E valid to first order follows immediately from eqs. (121),(122) and (123) plus the Hamiltonian constraint $E_1 = 0$, leading to

$$E = -p_1^2 \frac{v_1(x)}{t^{2/3}} - \alpha_1^2 \frac{w_1(x) - v_1(x)}{x^2 t^{2/3}} \tag{129}$$

where $t = t(E_1 = 0, p_1, \tau l1)$, $x = x(E_1 = 0, p_1, \alpha_1, \rho_1, \tau_1)$.

2. Corrections to p

The value of the old momentum p follows from

$$p = \frac{\partial W_1}{\partial \rho} = p_1 - p_1^2 \frac{\partial \mathcal{G}_1}{\partial \rho} - \alpha_1^2 \frac{\partial \mathcal{F}_1}{\partial \rho} \tag{130}$$

Eqs. (127) and (128) lead to

$$\frac{\partial \mathcal{G}_1}{\partial \rho} = \frac{1}{2p_1} \int\limits_{t_{today}}^{t} \frac{dt}{\sqrt{1 + E_1 \frac{t'^{4/3}}{p^2}}} v_1^{'}(x) \frac{dx}{d\rho} \tag{131}$$

$$\frac{\partial \mathcal{F}_1}{\partial \rho} = \frac{1}{2p_1} \int\limits_{t_{today}}^{t} \frac{dt}{\sqrt{1 + E_1 \frac{t'^{4/3}}{p^2}}} \left[\frac{w_1(x) - v_1(x)}{x^2} \right]^{'} \frac{dx}{d\rho} \tag{132}$$

Setting $E_1 = 0$, we have from the zeroth order

$$\frac{\partial x}{\partial \rho} = \frac{\rho - 3t^{1/3}}{v_0^2 \sqrt{\frac{(\rho - 3t^{1/3})^2}{v_0^2} + \frac{\alpha^2}{p^2}}} = \frac{\rho - 3t^{1/3}}{v_0^2 x} = \frac{\sqrt{p_1^2 x^2 - \alpha_1^2}}{v_0 p_1 x} \tag{133}$$

and consequently

$$\begin{aligned} p = p_1 - \frac{p_1}{2} \int\limits_{t_{today}}^{t} \frac{dt}{\sqrt{1 + E_1 \frac{t'^{4/3}}{p_1^2}}} v_1^{'}(x) \frac{\sqrt{p_1^2 x^2 - \alpha_1^2}}{v_0 p_1 x} - \\ - \frac{\alpha_1^2}{2p_1} \int\limits_{t_{today}}^{t} \frac{dt}{\sqrt{1 + E_1 \frac{t'^{4/3}}{p_1^2}}} \left[\frac{w_1(x) - v_1(x)}{x^2} \right]^{'} \frac{\sqrt{p_1^2 x^2 - \alpha_1^2}}{v_0 p_1 x} \end{aligned} \tag{134}$$

3. Corrections to α

The value of the old momentum α follows from

$$\alpha = \frac{\partial W_1}{\partial \chi} = \alpha_1 \tag{135}$$

a result that could be guessed due to the fact that the perturbations do not depend on the angles.

4. Corrections to ρ and τ

The new coordinates are

$$\rho_1 = \frac{\partial W_1}{\partial p_1} = \rho - 2p_1 G_1 - p_1^2 \frac{\partial \mathcal{G}_1}{\partial p_1} - \alpha_1^2 \frac{\partial \mathcal{F}_1}{\partial p_1} \tag{136}$$

At $E_1 = 0$ we have from eqs. (127) and (128) that

$$\frac{\partial \mathcal{G}_1}{\partial p_1} = -\frac{1}{2p_1^2}\int\limits_{\tau_{today}}^{\tau} dt\, v_1(x) + \frac{1}{2p_1}v_1(x)\frac{\partial t}{\partial p_1} + \frac{1}{2p_1}\int\limits_{\tau_{today}}^{\tau} dt\, v_1^{'}(x)\frac{\partial x}{\partial p_1} \tag{137}$$

$$\begin{aligned}\frac{\partial \mathcal{F}_1}{\partial p_1} = &-\frac{1}{2p_1^2}\int\limits_{\tau_{today}}^{\tau} dt' \left(\frac{w_1(x)-v_1(x)}{x^2}\right) + \frac{1}{2p_1}\left(\frac{w_1(x)-v_1(x)}{x^2}\right)\frac{\partial t}{\partial p_1} + \\ &+\frac{1}{2p_1}\int\limits_{\tau_{today}}^{\tau} dt' \frac{d}{dx}\left(\frac{w_1(x)-v_1(x)}{x^2}\right)\frac{\partial x}{\partial p_1}\end{aligned} \tag{138}$$

¿From the zeroth order, if $E = 0$, we have that $t = (10p\tau/3)^{3/5}$, so that $\partial t/\partial p = 3t/5p_1$. Also from this order

$$x = \sqrt{\frac{\left(\rho - 3t^{1/3}\right)^2}{v_0^2} + \frac{\alpha^2}{p^2}} \rightarrow \frac{\partial x}{\partial p} = \frac{-\alpha^2}{p^3\sqrt{\frac{\left(\rho-3t^{1/3}\right)^2}{v_0^2} + \frac{\alpha^2}{p^2}}} = \frac{-\alpha_1^2}{p_1^3 x} \tag{139}$$

Therefore we have

$$\begin{aligned}\rho = \rho_1 &+ \frac{1}{2}\int\limits_{\tau_{today}}^{\tau} dt\, v_1(x) + \frac{3}{10}v_1(x)t + \frac{3}{10}\frac{\alpha_1^2}{p_1^2}\left(\frac{w_1(x)-v_1(x)}{x^2}\right)t - \\ &-\frac{\alpha_1^2}{2p_1^2}\int\limits_{t_{today}}^{t} dt\frac{w_1(x)-v_1(x)}{x^2} - \frac{\alpha_1^2}{2p_1^2}\int\limits_{\tau_{today}}^{\tau} dt\frac{v_1^{'}(x)}{x} - \\ &-\frac{\alpha_1^4}{2p_1^4}\int\limits_{\tau_{today}}^{\tau} dt\left(\frac{w_1^{'}(x)-v_1^{'}(x)}{x^3}\right) + \frac{\alpha_1^4}{p_1^4}\int\limits_{t_{today}}^{t} dt\frac{w_1(x)-v_1(x)}{x^4}\end{aligned} \tag{140}$$

In order to compute the integrals, it is convenient to parametrize the null geodesic with the coordinate x, from eq. (115) we get

$$dt = -\left\{1 + \frac{v_0}{3pt_{today}}\left[\sqrt{p_1^2x_T^2 - \alpha_1^2} - \sqrt{p_1^2x^2 - \alpha_1^2}\right]\right\}^2 \frac{v_0 t_{today}^{2/3} p_1 x}{\sqrt{p_1^2x^2 - \alpha_1^2}}dx \tag{141}$$

Therefore the corrections computed above read

$$p = p_1 + \frac{p_1}{2}t_{today}^{2/3}v_1(x) + \frac{\alpha_1^2}{2p_1}t_{today}^{2/3}\frac{(w_1(x)-v_1(x))}{x^2} \tag{142}$$

$$\begin{aligned}\rho = \rho_1 &- t_{today}^{2/3}p_1 v_0\int\limits_0^x dx\,\frac{x}{\sqrt{p^2x^2-\alpha^2}}v_1(x) + \frac{3t}{10}v_1(x) + \frac{3\alpha_1^2}{10p_1^2}t\frac{w_1(x)-v_1(x)}{x^2} + \\ &+\frac{\alpha_1^2}{2p_1}t_{today}^{2/3}v_0\int\limits_0^x \frac{dx}{\sqrt{p^2x^2-\alpha^2}}v_1'(x) + \frac{\alpha_1^2}{p_1}t_{today}^{2/3}v_0\int\limits_0^x \frac{dx}{\sqrt{p^2x^2-\alpha^2}}\left(\frac{w_1(x)-v_1(x)}{x}\right) + \\ &\frac{\alpha_1^4}{2p_1^3}t_{today}^{2/3}v_0\int\limits_0^x \frac{dx}{\sqrt{p^2x^2-\alpha^2}}\frac{w_1'(x)-v_1'(x)}{x^2} - \frac{\alpha_1^4}{p_1^3}t_{today}^{2/3}v_0\int\limits_0^x \frac{dx}{\sqrt{p^2x^2-\alpha^2}}\frac{w_1(x)-v_1(x)}{x^3}\end{aligned} \tag{143}$$

where x is to be read from the leading order and where $v_1(x)$ and $w_1(x)$ are given by eq. (118).

Now we are ready to evaluate the next to leading order corrections to the redshift and $F^{1/3}(x)$. For the redshift we have to replace eqs. (129) and (142) into eq. (107), keeping only the leading and next to leading order terms. We get

$$P_t = \frac{p_1}{t^{2/3}} \left\{ 1 - \frac{\eta_{today}^2}{20a^2} \xi^{''}(x) \left(1 - \frac{t^{2/3}}{t_{today}^{2/3}} \right) - \frac{\eta_{today}^2}{20a^2} x_T^2 \sin^2\varphi \left[\frac{\xi^{'}(x)}{x^3} - \frac{\xi^{''}(x)}{x^2} \right] \left(1 - \frac{t^{2/3}}{t_{today}^{2/3}} \right) \right\} \tag{144}$$

where we have used $\alpha_1 = p_1 x_T \sin\varphi$.

In order to evaluate $F^{1/3}(x)$ we recall expression (106) and replace in it eqs. (143) and (129) obtaining

$$F^{1/3}(x) = (12 t_{today})^{1/3} \sqrt{\left[1 + \frac{x_T}{x_0}\cos\varphi - \frac{t^{1/3}}{t_{today}^{1/3}} + I(x) \right]^2 + \frac{x_T^2}{x_0^2}\sin^2\varphi} \tag{145}$$

where

$$I(x) = \frac{\eta_{today}}{20}\sqrt{\frac{10\gamma}{3a}} \int_{x_T}^{x} \frac{x\,dx}{\sqrt{x^2 - x_T^2\sin^2\varphi}} \xi^{''}(x) - \tag{146}$$

$$-\frac{\eta_{today}}{20}\sqrt{\frac{10\gamma}{3a}} x_T^2 \sin^2\varphi \int_{x_T}^{x} \frac{dx}{\sqrt{x^2 - x_T^2\sin^2\varphi}} \xi^{'''}(x) -$$

$$-\frac{\eta_{today}}{20}\sqrt{\frac{10\gamma}{3a}} x_T^2 \sin^2\varphi \int_{x_T}^{x} \frac{dx}{\sqrt{x^2 - x_T^2\sin^2\varphi}} \left[\frac{\xi^{'}}{x^2} - \frac{\xi^{''}(x)}{x} \right] + \tag{147}$$

$$+\frac{\eta_{today}}{20}\sqrt{\frac{10\gamma}{3a}} x_T^4 \sin^4\varphi \int_{x_T}^{x} \frac{dx}{\sqrt{x^2 - x_T^2\sin^2\varphi}} \left[3\frac{\xi^{'}}{x^4} - 3\frac{\xi^{''}(x)}{x^3} + \frac{\xi^{'''}}{x^2} \right]$$

In both equations (144) and (145) we have imposed as boundary conditions $P_t = 1$ and $F^{1/3}(x) = (12t_{today})^{1/3} x_T/x_0$ at $x = x_T$, $t = t_{today}$.

As earlier, our formulae are valid for any choice of $\xi(x)$.

VI. AN EXAMPLE: SCALE - INVARIANT PRIMORDIAL PERTURBATIONS

In order to apply our developments to a concrete case, we must now specified ζ_k in Eq. (16). In the absence of a specific model for the spectrum of a primeval fluctuations, it is often assumed that, at horizon crossing $|\zeta_k| = \beta k^{(n-4)/2}$. On a $t = const.$ surface the corresponding expression is

$$|\zeta_k| = \beta k^{(n-3)/2} \tag{148}$$

where n is known as the spectral index. This index is determined by the equation of state $\omega = p/\rho$ at $50-60$ e-folds before the end of Inflation, and its expression for scalar perturbations is $n = 1 - 3(1+\omega) + d\ln(1+\omega)/d\ln k$, where k is the wavenumber [26]. Strict exponential (de Sitter) expansion corresponds to $n = 1$, the so-called Harrison - Zel'dovich spectrum. But in any realistic inflationary model, the expansion rate must slow down near the end of Inflation in order to return to FRW expansion, and in this case $n \neq 1$ is expected [26]. The specification of the spectrum is concluded by also fixing the phases of the amplitudes for the different modes, which in inflationary models are usually random Gaussian variables; for simplicity, we shall disregard this, assuming $|\zeta_k| = \zeta_k$ instead. Of course, applying our formulae to the more general case of nontrivial phases involves no difficulties of principle.

With the form (148), we have

$$\gamma = \beta k_0^{(n+3)/2} \tag{149}$$

and

$$\xi(x) = \frac{1}{x}\int_0^1 \frac{du}{u} u^{(n+1)/2} \sin ux \tag{150}$$
$$= -\frac{i}{(n+1)x}\left[{}_1F_1\left(\frac{n+1}{2};\frac{n+3}{2};ix\right) - {}_1F_1\left(\frac{n+1}{2};\frac{n+3}{2};-ix\right)\right]$$

where ${}_1F_1\left(\frac{n+1}{2};\frac{n+3}{2};\pm ix\right)$ are the degenerate hypergeometric function [27]. From eqs. (20) and (21) we see that in general $F(x)$ and $f^2(x)$ will be oscillatory functions of x. The effect of the perturbations at the beginning of the matter dominated epoch is a mass redistribution, and the creation of negative curvature regions ($f^2(x) > 1$), that will expand forever, and of positive curvature ones ($f^2(x) < 1$), that will eventually collapse. This distribution of matter will evolve non linearly from its initial state at t_0, its evolution being described by the Tolman equations. We normalize the spectrum by considering the amplitude of the anisotropy of the cosmic background radiation corresponding to a scale of the size of the horizon at decoupling ($\lambda_d \simeq 0.13\, Mpc$) [28].

$$k_d^3\zeta_{k_d} = \gamma_n k_d^{(n+3)/2} \equiv \gamma\left(\frac{k_d}{k_0}\right)^{(n+3)/2} \simeq 6.6\times 10^{-5} \tag{151}$$

where we used $k_d^3\zeta_{k_d} \simeq 3\Delta T/T$. By virtue of the linear relationship between the angles that the scales subtend today and their sizes, we have $k_d/k_0 \sim \theta_0/\theta_d$, where θ_d is the angle subtended today by a scale of the size of the horizon at decoupling, $\theta_d \sim 1.4^\circ - 2^\circ$, and θ_0 is the angle subtended today by the wavelength corresponding to k_0, $\theta_0 \sim 45' - 1^\circ$. For concreteness we choose $\theta_0 \sim 48'$, $\theta_d \sim 1.4^\circ$.

With the form (152) for $\xi(x)$ we have $a = 1/12$. If we take $k_0 \simeq 918hMpc^{-1}$ (which corresponds to a wavelength about six times bigger than the horizon size at equilibrium), $t_0 \simeq 5.65\times 10^{-4}h^{-1}Mpc$, $t_{today} \simeq 1998h^{-1}Mpc$ and $\gamma \simeq 2.02125\times 10^{-4}$, we get $\eta \simeq 1.78$ (cfr. eq. (43)).

The model becomes particularly simple when $n = 1$, whereby eq. (150) becomes

$$\xi(x) = \frac{1}{x^2}\left[1 - \cos x\right] \tag{152}$$

This function is to be used in expressions found in the previous sections.

A. Density Contrast, Number Counts and Luminosity Distance

In Fig. **2** we have plotted the density contrast $\Delta\delta/\bar\delta$ vs z for the computed second order. As the matching conditions tell us nothing about the sign of $f^2(x) - 1$, we chose it so that the perturbation evolved to form a void. We see that the profile is of non compensated kind. After the non linear evolution, the density contrast in the wall is $\Delta\delta/\bar\delta \simeq 0.06$ and in the center of the void $\Delta\delta/\bar\delta \simeq -0.44$. If we define the radius of the void as the distance between its center and the point where $\Delta\delta/\bar\delta = 0$ (which corresponds to $z \simeq 0.024$), we get $R \simeq 72h^{-1}Mpc$, taking into account the results of Fig. **4**.

In Fig. **3** we have plotted $\ln N(x)$ vs $\ln z$. Dotted line corresponds to an unperturbed Universe, full line to one with second order perturbation. We see that the number count increases with z as we move over the increasing density zone. For $z \simeq 0.05$ the number count begins to equal that for a FRW Universe.

In order to analyze the departures from the linear Hubble law due to the presence of the perturbations, we have plotted in Fig. **4** the z vs d_ℓ relationship, for a flat FRW Universe (dotted line) and for Tolman Universe (full line). Due to the small redshifts considered, the slope of the curves can be considered a measure of the Hubble constant, which in our units is $H_0 = 2$ (the true value of the Hubble constant being $H_0 = 2/3t_{today}$). We see that the slope of the curve corresponding to the perturbed Universe is smaller than the one corresponding to FRW, up to $z \simeq 0.013$. From there on the slope starts to increase to finally coincide with the unperturbed one as expected.

B. Anisotropies in the CMBR

In our Tolman Universe, we find two sources of CMBR anisotropies, one of which is due to the fluctuations in the density, $\Delta T/T \propto \Delta\delta/\bar{\delta}$ ($\Delta\delta/\bar{\delta}$ given by eq.(85) evaluated using eq. (115)). The origin of the other is that being the Earth away from the center of symmetry, the light that was emitted from different points of last scattering surface will reach the observer with different redshifts, depending on the angular position of the source point, i.e. we have a contribution to the CMBR anisotropies of the form $z(\theta) - \bar{\zeta}$, where $\bar{\zeta}$ is the mean redshift of the decoupling surface. We find that in our case the last contribution outweights the first one by a factor of 10^7: we calculated $\Delta\delta/\bar{\delta}$ and $z(\theta)-\bar{\zeta}$ and after substracting the dipolar contribution due to the movement of the Earth ($\Delta T/T \approx 10^{-3}$) we were left with a quadrupolar contribution of $\Delta T/T \approx 10^{-13}$ and $\Delta T/T \approx 10^{-6}$, respectively, for big angular scales. The fact that the contribution from the density fluctuations is so small is not surprising in view of the small cut-off k_0.

VII. DISCUSSION

In this paper we have developed an analytical method to study the non linear evolution of adiabatic perturbations in a matter dominated Universe. We build the initial profile of the fluctuation on a $t = const.$ surface (that corresponds to the beginning of the matter dominated era, when structure formation begins) as a superposition of all those modes whose wavenumber is smaller than a certain cut-off k_0. This cut-off may correspond to the mode that survived the Meszaros effect [15], Landau damping or free streaming [19].

¿From this point on we can follow the non linear evolution of the perturbations by means of the exact solution to Einstein equation for a pressureless, spherically symmetric Universe, namely the Tolman Universe. Since this solution is isotropic with respect to one point, the Tolman solution can be applied only to analyze the formation of a structure (for example a void or cluster) with the proper symmetry. This restriction in the applicability of the Tolman solution to the study of structure evolution is compensated by the advantages of having an exact solution at our disposal.

Formally, the basis of all calculations for observable quantities lays in the resolution of the equation for the null geodesics, which in general is not trivial and is carried out by numerical methods. In this work we presented a perturbative method to do this calculation analytically. We put our calculations in a physical framework by the way in which we build the perturbations. We carried out the calculations only to second order, which means a mild non linear evolution, but the extension to higher orders is straightforward.

In the last section we obtained a glimpse of the results to be obtained by assuming a scale free spectrum of initial perturbations. In view of the simplicity of the calculations involved, we find that our results give a remarkable approximation to the structure of some of the largest known voids, such as Bötes'.Quantitatively, our results are accurate only up to an order of magnitude (in fact, the Bötes void is surrounded by a wall whose density contrast is $\Delta\delta/\bar{\delta} \simeq 4$, and its radius is of about $30Mpc$ [29]). The difference, of course, could be reduced by a better choice of the cutoff wavelength and, most importantly, by carrying the computation to higher orders.

With regard to future work, the most important feature of the method we propose, besides its analytical and relatively simple character, is that it may be applied for any form of the primordial spectrum. Thus it becomes only a matter of plugging in one's favorite theory of primordial fluctuation generation (as reflected in the particular form of ζ_k (Eq. (16)) to easily obtain (rough) testable predictions of that model. While we have used here a (unrealistic) scale invariant spectrum for demonstration purposes, the calculation is equally simple with red, blue, or more sophisticated alternatives.

We therefore believe the methods we describe in this paper will be an useful tool in the delicate task of sorting between the manyfold fluctuation generation scenarios now available.

VIII. ACKNOWLEDGMENTS

It is a pleasure to thank N. Deruelle, D. Harari and M. Zaldarriaga for comments on earlier versions of this work, and C. El Hasi, A. Gangui, and S. Gonorazky for their help on carrying it to completion. This

work has been partially supported by Universidad de Buenos Aires, CONICET and Fundación Antorchas, and by the Commission of the European Communities under Contract CI1*-CJ94-0004.

IX. APPENDIX A: EXPANSION OF THE RADIAL TOLMAN FUNCTION

$$R(x,t) = \frac{12^{2/3}}{4} F^{1/3}(x) t^{2/3} \left\{ 1 + \frac{12^{2/3}}{20} \frac{\left[f^2(x)-1\right]}{F^{2/3}(x)} t^{2/3} - \frac{3}{2800} 12^{4/3} \frac{\left[f^2(x)-1\right]^2}{F^{4/3}(x)} t^{4/3} + \cdots \right\} \tag{153}$$

$$\frac{1}{R^2(x,t)} = \frac{4^2}{12^{4/3}} \frac{1}{F^{2/3}(x) t^{4/3}} \left\{ 1 - \frac{12^{2/3}}{10} \frac{[f^2(x)-1]}{F^{2/3}(x)} t^{2/3} + \frac{3}{1400} 12^{4/3} \frac{[f^2(x)-1]^2}{F^{4/3}(x)} t^{4/3} + \right. \tag{154}$$
$$\left. + \frac{3}{400} 12^{4/3} \frac{[f^2(x)-1]^2}{F^{4/3}(x)} t^{4/3} + \cdots \right\}$$

$$\begin{aligned} \frac{1}{R'^2(x,t)} = \frac{4^2}{12^{4/3}} \frac{1}{\left[F^{1/3}(x)\right]'^2 t^{4/3}} & \left\{ 1 - \frac{1}{10} \frac{12^{2/3}}{\left[F^{1/3}(x)\right]'} \left[\frac{f^2(x)-1}{F^{1/3}(x)} \right]' t^{2/3} + \right. \\ & + \frac{3}{400} \frac{12^{4/3}}{\left[F^{1/3}(x)\right]'^2} \left[\frac{f^2(x)-1}{F^{1/3}(x)} \right]'^2 t^{4/3} + \\ & \left. + \frac{3}{1400} \frac{12^{4/3}}{\left[F^{1/3}(x)\right]'} \left[\frac{\left(f^2(x)-1\right)^2}{F(x)} \right]' t^{4/3} - \cdots \right\} \end{aligned} \tag{155}$$

X. APPENDIX B: EVALUATION OF Ω

We evaluate Ω today as follows. In a neighborhood of the origin, the evolution of the Universe is nearly like a FRW one, therefore

$$\Omega = 1 + \frac{\mathcal{K}}{H^2 a^2(t)} = 1 + \frac{\mathcal{K}}{\dot{a}^2(t)} \tag{156}$$

where H is the Hubble constant, $a(t)$ the expansion factor of the Universe and $\mathcal{K}$ the curvature. We have to extract the expressions for $\dot{a}(t)$ and $\mathcal{K}$ from the Tolman equations (14), (8) and (9). We define

$$\mathcal{K} \equiv -\frac{1}{2} \frac{d^2 f^2(x)}{dx^2} \mid_{x=0} \tag{157}$$

from where we get $\mathcal{K} = (10/3)\gamma\xi''(0) \simeq -5 \times 10^{-5}$. To obtain $\dot{a}(t)$ we calculate $\dot{R} = (dR/d\eta)(d\eta/dt)$ from Eqs. (8) and (9) obtaining

$$\dot{R}(x,\eta) = \sqrt{f^2(x)-1} \frac{\sinh\eta}{\cosh\eta - 1} = \sqrt{\frac{10}{3}\gamma a} x \frac{\sinh\eta}{\cosh\eta - 1} \tag{158}$$

where $a = |\xi''(0)|$. We therefore have

$$\dot{a}(t) = \sqrt{\frac{10}{3}\gamma a} \frac{\sinh\eta}{\cosh\eta - 1} \tag{159}$$

Replacing Eq. (159) in Eq. (156) we get

$$\Omega = 1 - \tanh^2\left(\frac{\eta}{2}\right) \tag{160}$$

Evaluating η from Eq. (9) with $t = t_{today}$ we get $\Omega_{today} \simeq 0.49$ at $x = 0$.

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FIGURE CAPTIONS

Fig. 1: Angle subtended by the null geodesics that arrive at the observer's position x_T.

Fig. 2: Density contrast $\Delta\delta/\bar{\delta}$ vs redshift z, to second order in the perturbative development. The profile corresponds to a non compensated void.

Fig. 3: $\ln N$ vs $\ln z$, to second order in the perturbative development. Dashed line corresponds to flat FRW Universe, full line to Tolman Universe

Fig. 4: Redshift z vs luminosity distance dl, to second order in the perturbative development. Due to the small redshifts considered, the slope of the curve is a measure of the Hubble constant. Dashed line corresponds to flat FRW Universe, full line to Tolman Universe.

Fig. 1

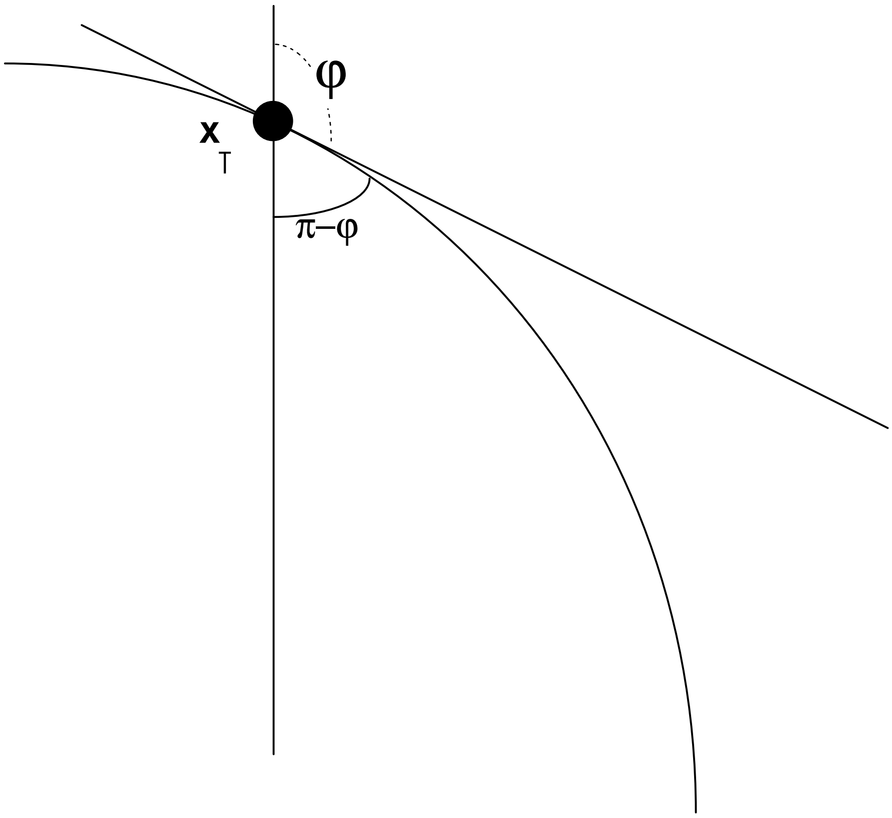

Fig. 2: Density Contrast Vs Redshift

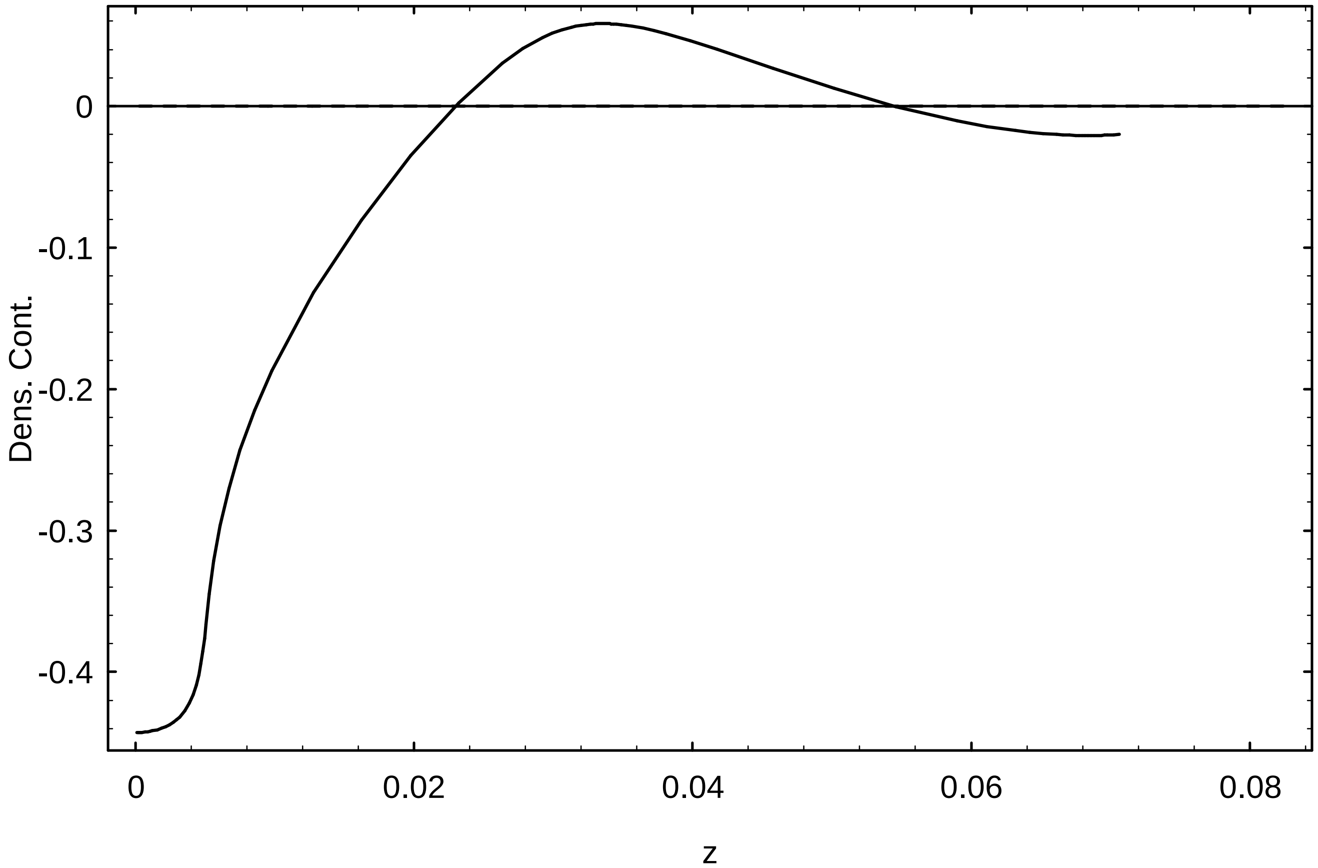

Fig. 3:Ln(N) Vs Ln(z)

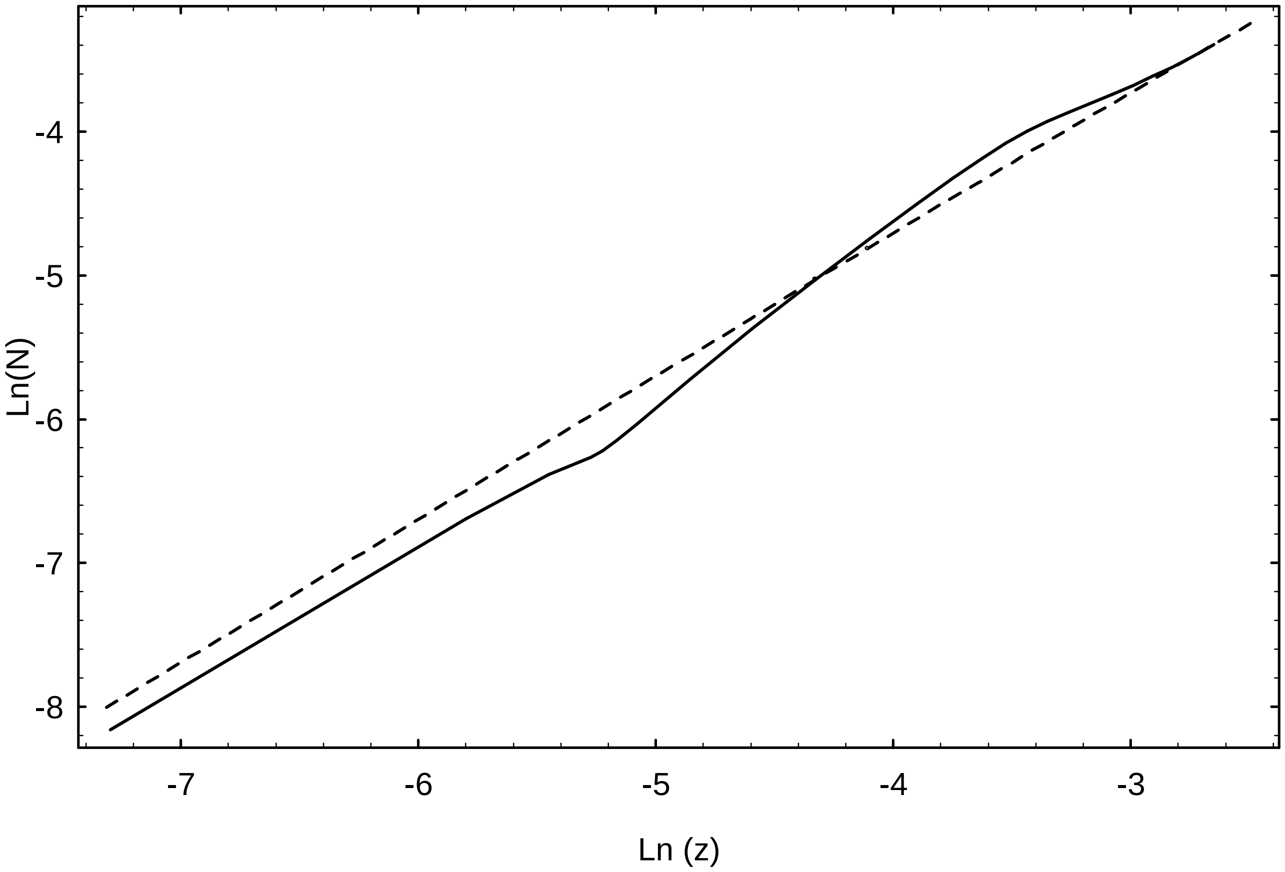

Fig. 4:z vs dl